An Additive Structure of Viterbi Semirings

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Abstract: In this paper, we study on additive and multiplicative properties of viterbi semirings. The modern interest in semirings arises primarily from fields of Applied Mathematics such as Optimization theory, the theory of discrete-event dynamical systems, automata theory and formal language theory, as well as from the allied areas of theoretical computer science and theoretical physics and the questions being asked is, for the most part, motivated by applications. This paper deals with some definitions, which are needed for the study of main results of this paper. We also discuss the additive properties of viterbi semirings. Here, we established that let (S, +, •) be a viterbi semiring, then S satisfies the condition $a^2 = a + a^2$, for all `a' in S, if and only if (S, •) is a band.

Index Terms: Positive Rational Domain (PRD); Integral Multiple Property (IMP); Band; Quasi commutative; Rectangular Band.; Multiplicatively subidempotent.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 20M10, 16Y60.

1. INTRODUCTION

The study of rings, which are special semirings shows that the multiplicative structures are quite independent though their additive structures are abelian groups. However in semirings it is possible to derive the additive structures from their special multiplicative structures and vice versa. The developments of semirings and ordered semirings in this direction require semigroup techniques. It is well known that if the multiplicative structure of an ordered semiring is a rectangular band then its additive structure is a band. P.H. Karvellas [21] has proved that if the multiplicative structure of a semiring is a regular semigroup and if the additive structure is an inverse semigroup, then the additive structure is commutative.

There is considerable impact of semigroup theory and semiring theory on the development of ordered semirings both in theory and applications, which are Akin to ordered rings and ordered semirings. In this direction the works of H. J. weinert [47], M.Satyanarayana, [33, 35, 36, 37, 38, 39]J. Hanumanthachari, K.Venuraju and H.J Weinert [12], M.Satyanarayana, J.Hanumanthachari and D. Umamaheswara Reddy [40,41], K.P. Shum and C.S.Hoo [44], K.P.Shum and C.Y. Hung [45], Kehayopulu [22], G.Therrin, H. Jurgensen, H.J. Shyr, Gerand Lallment and U. Zimmermann are to be worth mentioning.

The viterbi semirings are a sort of cross between arithmetic and ordering. They are, however, rather common in computer science. They tend to arise whenever we use dynamic programming in order to find the "best" of some set of things: e.g., the most likely tag sequence, the shortest path, the longest path, etc. This is what part of speech tagging, matrix chain multiplication, and Dijkstra's algorithm are all about. The concept of viterbi Semiring is taken from the book of Jonathan S.Golan [3], entitled "Semirings and Affine Equations over Them: Theory and Applications".In this paper, we characterize viterbi semirings.

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2. PRELIMINARY DEFINITIONS

2.1 Semiring:

A triple (S, +, •) is said to be a semiring if S is a non - empty set and "+, •" are binary operations on S satisfying that

(i) (S, +) is a semigroup
(ii) (S, •) is a semigroup
(iii) a(b + c) = ab + ac and (b + c)a = ba + ca, for all a, b, c in S.

2.2 Examples Of Semiring:

(i) The set of natural numbers under the usual addition, multiplication
(ii) Every distributive lattice (L, $\wedge$, $\vee$).
(iii) Let S = {a, b} with the operations given by the following tables:

+	a	b
a	a	b
b	b	b

•	a	b
a	b	b
b	b	b

Then (S, +, •) is a semiring.

2.3 Multiplicatively Subidempotent :

An element `a' of a semiring S is multiplicatively subidempotent if and only if $a + a^2 = a$ and S is multiplicatively subidempotent if and only if each of its elements is multiplicatively subidempotent.

2.4 Viterbi Semiring :

A viterbi semiring is a semiring in which S is additively idempotent and multiplicatively subidempotent.

i.e., $a + a = a$ and $a + a^2 = a$, for all 'a' in S.

2.5 Rectangular Band:
A semigroup (S, +) satisfying the identity $x = x + y + x$, for all x, y in S , is a rectangular band.

2.6 Band:
A semigroup (S, •) is said to be a band if $a^2 = a$, for all 'a' in S.

2.7 Integral Multiple Property:
A semiring (S, +, •) is said to satisfy the Integral Multiple Property (IMP) if $a^2 = na$, for all a in S where the positive integer n depends on the element a.

2.8 Quasi Commutative:
A semigroup (S, •) is said to satisfy quasi commutative if $ab = b^m a$ for some integer $m \geq 1$.

2.9 Positive Rational Domain:
A semiring (S, +, •) is said to be a Positive Rational Domain (PRD) if and only if (S, •) is an abelian group.

3. MAIN RESULTS
In this section, the properties of viterbi semirings are studied. We proved that (S, +, •) be a viterbi semiring containing multiplicative identity 1 and if (S, •) is left cancellative, then $1 + a = 1$, for all 'a' in S.

3.1 Theorem : Let (S, +, •) be a viterbi semiring containing multiplicative identity 1. If (S, •) is left cancellative, then $1 + a = 1$, for all 'a' in S.

Proof: Consider $a + a^2 = a$, for all 'a' in S
$\Rightarrow a.1 + a^2 = a.1$
$\Rightarrow a\,(1 + a) = a.1$
$\Rightarrow 1 + a = 1$ ($\because$ (S, +) is left cancellative)
$\therefore$ $1 + a = 1$, for all 'a' in S.

3.2 Theorem : Let (S, +, •) be a viterbi semiring containing multiplicative identity 1. If (S, +) is right cancellative, then (S, +) is a rectangular band.

Proof: Since (S, +, •) is a viterbi semiring
We have $a + a = a$ and $a + a^2 = a$, for all 'a' in S
Let $a, b \in S$
Then $a + b \in S$ ($\because$ (S, +) is a semigroup)
Also $a + b + a + b = a + b$
($\because$ S is a viterbi semiring)
$\Rightarrow a + b + a = a$ ($\because$ (S, +) is right cancellative)
$\therefore$ (S, +) is a rectangular band.

3.3 Theorem : Let (S, +, •) be a viterbi semiring. Then S satisfies the condition $a^2 = a + a^2$, for all 'a' in S, if and only if (S, •) is a band.

Proof: Since (S, +, •) is a viterbi semiring
$\Rightarrow a + a = a$ and $a + a^2 = a$, for all 'a' in S
Suppose $a^2 = a + a^2$
$\Rightarrow a^2 = a$ ($\because a + a^2 = a$)
$\therefore$ (S, •) is a band.
Conversely,
Suppose (S, •) is a band
i.e., $a^2 = a$, for all 'a' in S
$\Rightarrow a^2 = a + a^2$ ($\because a + a^2 = a$)
$\therefore$ $a^2 = a + a^2$, for all 'a' in S.

3.4 Theorem: Let (S, +, •) be a viterbi semiring. If S satisfies Integral Multiple Property (IMP), then the following are true
(i) (S, •) is a band
(ii) If (S, •) is quasi commutative, then (S, •) is commutative

Proof: By hypothesis, (S, +, •) is a viterbi semiring
Hence $a + a = a$ and $a + a^2 = a$, for all 'a' in S
(i) Consider $a + a = a$
$\Rightarrow 2a = a$
$\Rightarrow 2a + a = a + a$
$\Rightarrow 3a = 2a = a$ ($\because 2a = a$)
$\Rightarrow 3a = a$
$\Rightarrow na = a$, for all $n > 1$
Since S satisfies IMP, $a^2 = na$
$\Rightarrow a^2 = a$
$\therefore$ (S, •) is a band.
(ii) Suppose (S, •) is quasi commutative
i.e., $ab = b^m a$, for some positive integer $m \geq 1$
But every element in (S, •) is an idempotent
($\because$ By using(i))
$\Rightarrow b^m = b$, for some positive integer $m \geq 1$
$\therefore$ $ab = ba$, for every b in S
Hence (S, •) is commutative.

3.5 Theorem : Let (S, +, •) be a Positive Rational Domain (PRD) viterbi semiring, then $b + ab = b$, for all b in S.

Proof: By hypothesis, (S, +, •) is a viterbi semiring
$\Rightarrow a + a = a$ and $a + a^2 = a$, for all 'a' in S
Consider $a + a^2 = a$
$\Rightarrow a.1 + a.a = a.1$
$\Rightarrow a\,(1 + a) = a.1$
$\Rightarrow (1 + a) = 1$
$\Rightarrow (1 + a)\,b = 1.b$
$\Rightarrow b + ab = b$
$\therefore$ $b + ab = b$, for all a, b in S

4 CONCLUSION

ISSN 2229-5518

1. The multiplicative structure of the viterbi semiring (S, +, •) with identity have the additive structure.
2. A Viterbi semiring (S, +, •) contains multiplicative identity 1 and (S, +) is right cancellative, then the additive semigroup (S, +) become a rectangular band.
3. The Viterbi semiring (S, +, •) satisfies the condition $a^2 = a + a^2$, for all \`a' in S, then the multiplicative semigroup (S, •) is a band. The converse is also true.
4. A Viterbi semiring (S, +, •) satisfies Integral Multiple Property (IMP), then the multiplicative semigroup (S, •) is a band.
5. We introduce positive rational domain (PRD) in a viterbi semiring (S, +, •), then b + ab = b, for all b in S.

5. REFERENCES

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